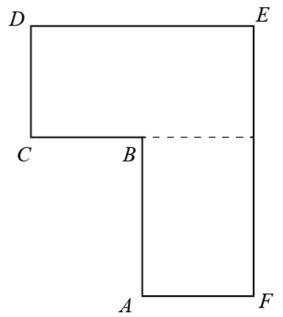
Review Exercise 1

1 a



Let A be the origin and let AF lie on the positive x-axis.

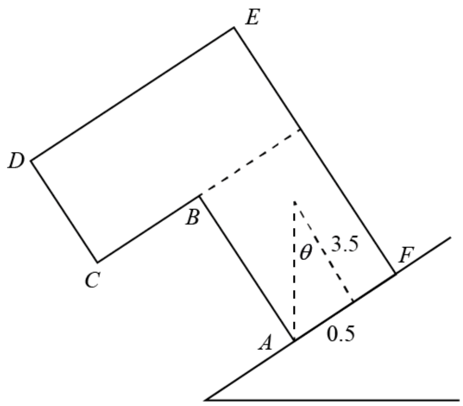
$$16\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 8\begin{pmatrix}1\\ 2\end{pmatrix} + 8\begin{pmatrix}0\\ 5\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{16}\begin{pmatrix}8\\ 56\end{pmatrix}$$

$$= \begin{pmatrix}0.5\\ 3.5\end{pmatrix}$$

So the centre of mass lies 0.5 cm from AB and 3.5 cm from AF

b

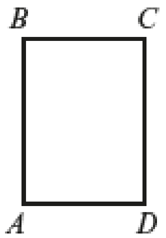


$$\tan\theta = \frac{0.5}{3.5}$$

$$\theta = 8.1301...$$

$$= 8.13° \text{ (3 s.f.)}$$

2



Let A be the origin and let AD lie on the positive x-axis.
The centre of mass of the lamina is at the point (2, 3)
Then the y-coordinate of the centre of mass of successive laminas would be;
3, 9, 15, …
Let n be the number of laminas that can be placed on top of each other.
When $n = 1$

$$\tan 10 = 0.176... < \frac{2}{3}$$

Therefore the lamina will not topple.
number of laminas that can be placed on top of each other.
When $n = 2$

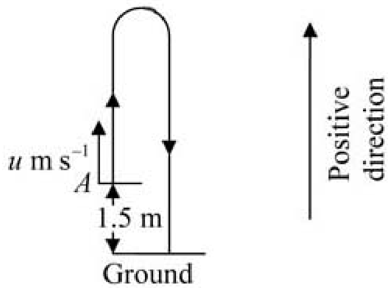
$$\tan 10 = 0.176... < \frac{2}{9}$$

Therefore the lamina will not topple.
When $n = 3$

$$\tan 10 = 0.176... > \frac{2}{15}$$

Therefore the lamina will topple.

3 a



From A to the greatest height, taking upwards as positive:

$v = 0,\ a = -9.8,\ s = 25.6,\ u = ?$

$$v^2 = u^2 + 2as$$

$$0^2 = u^2 + 2 \times (-9.8) \times 25.6$$

$$u^2 = 2 \times 9.8 \times 25.6 = 501.76$$

$$u = \sqrt{501.76} = 22.4, \text{ as required.}$$

3 b $u = 22.4,\ s = -1.5,\ a = -9.8,\ t = T$

$s = ut + \frac{1}{2}at^2$

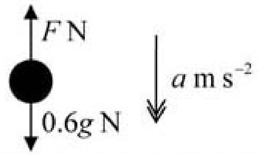
$-1.5 = 22.4T + \frac{1}{2}(-9.8)T^2$

$4.9T^2 - 22.4T - 1.5 = 0$

$$T = \frac{22.4 + \sqrt{(-22.4)^2 - 4\times 4.9\times -1.5}}{2\times 4.9}$$

$= 4.637... = 4.64$ (3 s.f.)

c



Find the speed of the ball as it reaches the ground:

$u = 22.4,\ s = -1.5,\ a = -9.8,\ v = ?$

$v^2 = u^2 + 2as = 22.4^2 + 2\times(-9.8)\times(-1.5) = 531.16$

Find the deceleration as the ball sinks into the ground:

$u^2 = 531.16,\ v = 0,\ s = 0.025,\ a = ?$

$v^2 = u^2 + 2as \Rightarrow 0^2 = 531.16 + 2\times a\times 0.025$

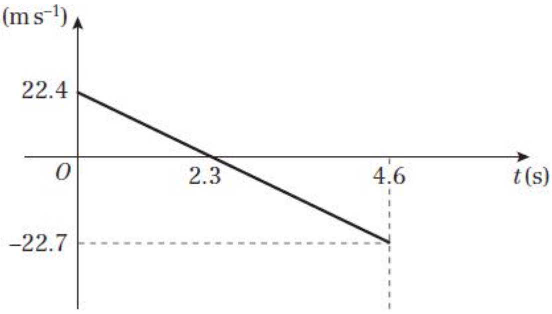
$a = -\frac{531.16}{0.05} = -10\,623.2$

$F = ma$

$0.6g - F = 0.6\times(-10\,623.2)$

$F = 0.6g + 0.6\times 10\,623.2 = 6380$ (3 s.f.)

d



e Consider air resistance during motion under gravity.

4 **a** $u_y = 0$, $a_y = 9.8 \text{ m s}^{-2}$, $s_y = 0.8 \text{ m}$

Using $s = ut + \frac{1}{2}at^2$ gives:

$0.8 = (0)t + \frac{1}{2}(9.8)t^2$

$4.9t^2 = 0.8$

$t = \frac{2\sqrt{2}}{7} \text{ s}$

$t = 0.404 \text{ s (3 s.f.)}$

b $u_x = 2 \text{ m s}^{-1}$, $a_x = 0$, $t = \frac{2\sqrt{2}}{7} \text{ s}$

Using $s = ut + \frac{1}{2}at^2$ gives:

$s = 2\left(\frac{2\sqrt{2}}{7}\right) + \frac{1}{2}(0)\left(\frac{2\sqrt{2}}{7}\right)^2$

$= \frac{4\sqrt{2}}{7} \text{ m}$

$t = 0.808 \text{ m (3 s.f.)}$

5 **a** Let the horizontal distance travelled be x.
By Pythagoras' theorem:

$x = \sqrt{40^2 - 20^2}$

$= 20\sqrt{3} \text{ m}$

$s_y = 20 \text{ m}$, $a_y = 9.8 \text{ m s}^{-2}$, $u_y = 0$,

Using $s = ut + \frac{1}{2}at^2$ gives:

$20 = (0)t + \frac{1}{2}(9.8)t^2$

$20 = (0)t + \frac{1}{2}(9.8)t^2$

$t = \frac{10\sqrt{2}}{7} \text{ s}$

$t = \frac{10\sqrt{2}}{7}, a_x = 0, s_x = 20\sqrt{3}$

$20\sqrt{3} = u\left(\frac{10\sqrt{2}}{7}\right) + \frac{1}{2}(0)\left(\frac{10\sqrt{2}}{7}\right)^2$

$u_x = 7\sqrt{6} \text{ m s}^{-1}$

$u_x = 17.1 \text{ m s}^{-1} \text{(3 s.f.)}$

b The ball as a projectile has negligible size and is subject to negligible air resistance.
Free fall acceleration remains constant during flight of ball.

6 a $u_y = 150\sin 10 \text{ m s}^{-1}$, a_y = -9.8 m s^{-2}, $v_y = 0$

Using $v = u + at$ gives:

$0 = 150\sin 10 - 9.8t$

$t = 2.657...$

$= 2.66$ s (3 s.f.)

b $u_x = 150\cos 10 \text{ m s}^{-1}$, $a_x = 0$, $t = 2.657\ldots$ s

Using $s = ut + \frac{1}{2}at^2$ gives:

$s = (150\cos 10)(2.657...) + \frac{1}{2}(0)(2.657...)^2$

$= 392.625...$

$= 393$ m (3 s.f.)

7 a $u_y = 3u$, $a_y = -9.8 \text{ m s}^{-2}$, $s_y = -12$ m, $t = 3$ s

Using $s = ut + \frac{1}{2}at^2$ gives:

$-12 = 3u(3) + \frac{1}{2}(-9.8)(3)^2$

$-12 = 3u(3) + \frac{1}{2}(-9.8)(3)^2$

$9u = 32.1$

$u = \frac{107}{30}$

u = 3.57 ms^{-1} (3 s.f.)

7 b $u_x = 8u = \dfrac{428}{15}$, $a_x = 0$, $t = 3$ s

Using $s = ut + \frac{1}{2}at^2$ gives:

$$s = \left(\frac{428}{15}\right)(3) + \frac{1}{2}(0)(3)^2$$

$$= \frac{428}{5} \text{ m}$$

k = 85.6 m

c $u_y = \dfrac{107}{10}$ m, $a_y = -9.8$ m s^{-2}, $s_y = -30$ m

Using $v^2 = u^2 + 2as$ gives:

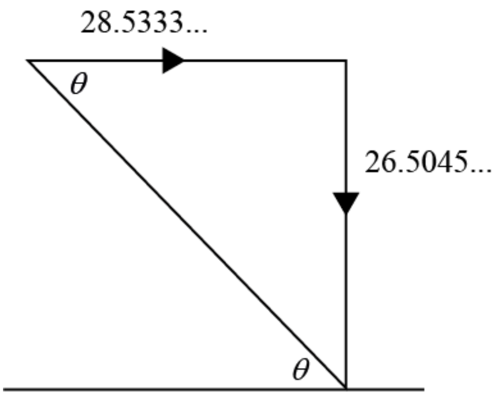
$$v_y^{\,2} = \left(\frac{107}{10}\right)^2 + 2(-9.8)(-30)$$

$$v_y^{\,2} = 702.49$$

$$v_y = \pm 26.5045....$$

$$v_y = -26.5045....$$

In the x-direction, $v_x = \dfrac{428}{15}$ 8



$$\tan\theta = \frac{26.5045..}{28.5333...}$$

$$\theta = 42.8889...$$

$$= 42.9° \text{ (3 s.f.)}$$

8 a $u_y = u\sin\alpha$, $a_y = -g$, $s_y = 0$

Using $s = ut + \frac{1}{2}at^2$ gives:

$$0 = u\sin\alpha t - \frac{1}{2}gt^2$$

$t = \dfrac{2u\sin\alpha}{g}$ as required.

8 **b** $u_x = u\cos\alpha$, $a_x = 0$, $s_x = R$, $t = \dfrac{2u\sin\alpha}{g}$

Using $s = ut + \frac{1}{2}at^2$ gives:

$$R = u\cos\alpha\left(\frac{2u\sin\alpha}{g}\right)$$

$$= \frac{u^2\cos\alpha\sin\alpha}{g}$$

$$= \frac{2u^2\sin 2\alpha}{g} \text{ as required}$$

c $R = \dfrac{u^2\sin 2\alpha}{g}$

$$\frac{dR}{d\alpha} = \frac{2u^2\cos 2\alpha}{g}$$

The maximum range occurs when $\dfrac{dR}{d\alpha} = 0$

$$\frac{2u^2\cos 2\alpha}{g} = 0$$

$2\alpha = 90^0$

$\alpha = 45^0$ as required

d $R = \dfrac{2u^2}{5g}$ and $R = \dfrac{u^2\sin 2\alpha}{g}$

$$\frac{2u^2}{5g} = \frac{u^2\sin 2\alpha}{g}$$

$$\sin 2\alpha = \frac{2}{5}$$

$2\alpha = 23.5781...$ or $2\alpha = 180 - 23.781... = 156.4218...$

So $\alpha = 11.8°$ or $\alpha = 78.2°$ (3 s.f.)

9 $a = 5 - 2t$

$v = \int a\mathrm{d}t = \int (5 - 2t)\mathrm{d}t$

$= 5t - t^2 + C$

When $t = 0,\ v = 6$

$6 = 0 - 0 + C \Rightarrow C = 6$

Hence

$v = 6 + 5t - t^2$

When P is at rest

$0 = 6 + 5t - t^2$

$t^2 - 5t - 6 = (t - 6)(t + 1) = 0$

$t = 6, -1$

$t > 0$

$\therefore t = 6$

P is at rest at $t = 6$ s

10 $v = 6t - 2t^2$

a Maximum value of velocity occurs when $a = 0$

$a = \frac{\mathrm{d}v}{\mathrm{d}t} = 6 - 4t$

Maximum velocity occurs at $t = \frac{3}{2}$ s

$v = \left(6 \times \frac{3}{2}\right) - 2\left(\frac{3}{2}\right)^2$

$v = 9 - \frac{9}{2} = \frac{9}{2}$

The maximum velocity is 4.5 ms^{-1}.

10 b When P returns to O, $s = 0$

$$s = \int v\,\mathrm{d}t = \int 6t - 2t^2\,\mathrm{d}t$$

$$s = 3t^2 - \frac{2}{3}t^3 + c$$

At $t = 0$, $s = 0$ so $c = 0$

$$0 = t^2\left(3 - \frac{2}{3}t\right)$$

$$t = 0 \text{ or } \frac{2}{3}t = 3$$

P returns to O after 4.5 s.

11 $v = 3t^2 - 8t + 5$

a When the particle is at rest, $v = 0$

$$0 = 3t^2 - 8t + 5$$

$$0 = 3\left(t^2 - \frac{8}{3}t + \frac{5}{3}\right)$$

$$0 = 3\left(t - \frac{3}{3}\right)\left(t - \frac{5}{3}\right)$$

(or by using quadratic equation formula)

P is at rest at 1 s and $\frac{5}{3}$ s.

b $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(3t^2 - 8t + 5)$

$a = 6t - 8$

$t = 4$

$a = (6 \times 4) - 8$

After 4 s, the acceleration of P is 16 ms^{-2}.

c Distance travelled in third second $= s_3$

$$s_3 = \int_2^3 v\,\mathrm{d}t = \int_2^3 3t^2 - 8t + 5\,\mathrm{d}t$$

$$s_3 = \left[t^3 - 4t^2 + 5t\right]_2^3$$

$$s_3 = [27 - 36 + 15] - [8 - 16 + 10]$$

$$s_3 = 6 - 2$$

The distance travelled in the third second is 4 m.

12 a $v = 6t - 2t^{\frac{3}{2}},\ t \geq 0$

$a = \dfrac{dv}{dt} = 6 - 3t^{\frac{1}{2}}\,\text{ms}^{-2}$

b $v = 6t - 2t^{\frac{3}{2}}$

$s = \int\left(6t - 2t^{\frac{3}{2}}\right)dt$

$= 3t^2 - \dfrac{4}{5}t^{\frac{5}{2}} + c$

When $t = 0$, $s = 0$, therefore:

$0 = 3(0)^2 - \dfrac{4}{5}(0)^{\frac{5}{2}} + c$

$c = 0$

$s = 3t^2 - \dfrac{4}{5}t^{\frac{5}{2}}\,\text{m}$

13 a $\mathbf{r} = \left(\dfrac{1}{3}t^3 + 2t\right)\mathbf{i} + \left(\dfrac{1}{2}t^2 - 1\right)\mathbf{j}$

$\boldsymbol{v} = \dfrac{d\mathbf{r}}{dt} = \left(t^2 + 2\right)\mathbf{i} + t\mathbf{j}\,\text{ms}^{-1}$

b When $t = 5$ s

$\dfrac{d\mathbf{r}}{dt} = \left((5)^2 + 2\right)\mathbf{i} + (5)\mathbf{j}$

$= (27\mathbf{i} + 5\mathbf{j})$

$\left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{27^2 + 5^2}$

$= 27.459$

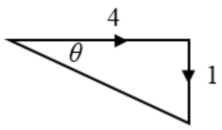
$= 27.5\ \text{ms}^{-1}$ (3 s.f.)

13 c $\frac{d\mathbf{r}}{dt} = (t^2 + 2)\mathbf{i} + t\mathbf{j}$

$\boldsymbol{a} = \frac{d^2\mathbf{r}}{dt^2} = 2t\mathbf{i} + \mathbf{j}$

When $t = 2$ s

$\frac{d^2\mathbf{r}}{dt^2} = 4\mathbf{i} + \mathbf{j}$



$\left|\frac{d^2\mathbf{r}}{dt^2}\right| = \sqrt{4^2 + 1^2}$

$= \sqrt{17}\ \text{ms}^{-2}$

$\tan\theta = \frac{1}{4}$

$\theta = 14.036...$

$= 14.0°$ (3 s.f.)

14.0° below the horizontal

14 a $\mathbf{r} = (4t^2 + 1)\mathbf{i} + (2t^2 - 3)\mathbf{j}$

$\boldsymbol{v} = \frac{d\mathbf{r}}{dt} = 8t\mathbf{i} + 4t\mathbf{j}$

When $t = 3$ s

$\frac{d\mathbf{r}}{dt} = 8(3)\mathbf{i} + 4(3)\mathbf{j}$

$= 24\mathbf{i} + 12\mathbf{j}\ \text{m s}^{-1}$

b $\boldsymbol{a} = \frac{d^2\mathbf{r}}{dt^2} = (8\mathbf{i} + 4\mathbf{j})\ \text{m s}^{-2}$ therefore the acceleration is constant.

15 $\mathbf{v} = -2t\mathbf{i} + 3\sqrt{t}\mathbf{j}$

$\mathbf{v} = -2t\mathbf{i} + 3t^{\frac{1}{2}}\mathbf{j}$

$\mathbf{s} = \int\left(-2t\mathbf{i} + 3t^{\frac{1}{2}}\mathbf{j}\right)\mathrm{d}t$

$= -t^2\mathbf{i} + 2t^{\frac{3}{2}}\mathbf{j} + c$

When $t = 0$, $\mathbf{s} = 2\mathbf{j}$

$2\mathbf{j} = -(0)^2\,\mathbf{i} + 2(0)^{\frac{3}{2}}\,\mathbf{j} + c$

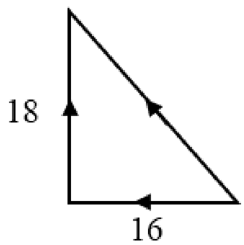
$c = 2\mathbf{j}$

$\mathbf{s} = -t^2\mathbf{i} + \left(2t^{\frac{3}{2}} + 2\right)\mathbf{j}$

When $t = 4$ s

$\mathbf{s} = -(4)^2\,\mathbf{i} + \left(2(4)^{\frac{3}{2}} + 2\right)\mathbf{j}$

$= -16\mathbf{i} + 18\mathbf{j}$



$|\mathbf{s}| = \sqrt{(-16)^2 + 18^2}$

$= 2\sqrt{145}$ m

16 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int\left(2t^2 - 3t^3\right)\mathbf{i} - 4(2t+1)\,\mathbf{j}\;\mathrm{d}t$

$\mathbf{v} = \left(t^2 - \frac{3}{4}t^4\right)\mathbf{i} - 4\left(t^2 + t\right)\mathbf{j} + c$

$t = 0 \Rightarrow \mathbf{v} = (3\mathbf{i} + \mathbf{j})$ m s^{-1}

$3\mathbf{i} + \mathbf{j} = 0\mathbf{i} - 4(0)\mathbf{j} + c$

$c = 3\mathbf{i} + \mathbf{j}$

$\Rightarrow \mathbf{v} = \left(t^2 - \frac{3}{4}t^4 + 3\right)\mathbf{i} - \left(4t^2 + 4t - 1\right)\mathbf{j}$

16 b If P is moving in the direction of **i**, the coefficient of **j** in the velocity vector is 0.

$0 = 4t^2 + 4t - 1$

$t = \dfrac{-4 \pm \sqrt{16 - (4 \times 4 \times (-1))}}{8}$

$t = \dfrac{-1 \pm \sqrt{2}}{2}$

The negative solution can be ignored as it is outside the range over which the equation applies.

P is moving in the direction of **i** after $\left(\dfrac{\sqrt{2}-1}{2}\right)$ s (0.207 s to 3 s.f.).

17 a $\mathbf{v} = \int \mathbf{a}\,dt = \int(-4t\mathbf{i} - 2\mathbf{j})\,dt$

$\mathbf{v} = -2t^2\mathbf{i} - 2t\mathbf{j} + c$

$t = 0 \Rightarrow \mathbf{v} = 8\mathbf{i}\text{ ms}^{-1}$

$8\mathbf{i} = 0\mathbf{i} - 0\mathbf{j} + c$

$c = 8\mathbf{i}$

$\Rightarrow \mathbf{v} = 2(4 - t^2)\mathbf{i} - 2t\mathbf{j}$

b When the windsurfer is moving due south, the coefficient of **i** in the velocity vector is 0.

$0 = 2(4 - t^2)$

$t^2 = 4$

$t = \pm 2$

The negative solution can be ignored as it is before the time the windsurfer starts to move.

When $t = 2$, $v = -2 \times 2\mathbf{j} = -4\mathbf{j}$

The windsurfer is moving due south after 2 s.

18 a $(8 + \lambda)m\begin{pmatrix}2\\k\end{pmatrix} = 3m\begin{pmatrix}4\\0\end{pmatrix} + 5m\begin{pmatrix}0\\-3\end{pmatrix} + \lambda m\begin{pmatrix}4\\2\end{pmatrix}$

$(8 + \lambda)\begin{pmatrix}2\\k\end{pmatrix} = \begin{pmatrix}12 + 4\lambda\\-15 + 2\lambda\end{pmatrix}$

$2(8 + \lambda) = 12 + 4\lambda$

$16 + 2\lambda = 12 + 4\lambda$

$2\lambda = 4$

$\lambda = 2$ as required.

b $10k = -15 + 4$

$k = -\dfrac{11}{10}$

19 $(2+x+y)M\begin{pmatrix}2\\4\end{pmatrix}=2M\begin{pmatrix}2\\5\end{pmatrix}+xM\begin{pmatrix}1\\3\end{pmatrix}+yM\begin{pmatrix}3\\1\end{pmatrix}$

$\begin{pmatrix}4+2x+2y\\8+4x+4y\end{pmatrix}=\begin{pmatrix}4+x+3y\\10+3x+y\end{pmatrix}$

$4+2x+2y=4+x+3y\Rightarrow x-y=0\Rightarrow x=y$ (1)

$8+4x+4y=10+3x+y\Rightarrow x+3y=2$ (2)

Substituting (1) into (2) gives:

$x+3x=2$

$x=\frac{1}{2}$

Therefore:

$y=\frac{1}{2}$

20 $0.6\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=0.1\begin{pmatrix}2\\-1\end{pmatrix}+0.2\begin{pmatrix}2\\5\end{pmatrix}+0.3\begin{pmatrix}4\\2\end{pmatrix}$

$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=\frac{1}{0.6}\begin{pmatrix}0.2+0.4+1.2\\-0.1+1.0+0.6\end{pmatrix}$

$=\frac{1}{0.6}\begin{pmatrix}1.8\\1.5\end{pmatrix}$

$=\begin{pmatrix}3\\2.5\end{pmatrix}$

Therefore the centre of mass lies at:

$(3\mathbf{i}+2.5\mathbf{j})$ m

21 a $(3+k)M\begin{pmatrix}3\\c\end{pmatrix}=2M\begin{pmatrix}6\\0\end{pmatrix}+M\begin{pmatrix}0\\4\end{pmatrix}+kM\begin{pmatrix}2\\-2\end{pmatrix}$

$\begin{pmatrix}9+3k\\3c+ck\end{pmatrix}=\begin{pmatrix}12+2k\\4-2k\end{pmatrix}$

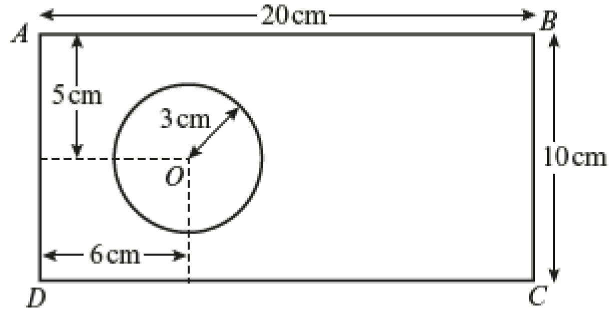
$9+3k=12+2k$

$k=3$ as required

b $3c+3c=-2$

$c=-\frac{1}{3}$

22 a



Let D be the origin and let DC lie on the positive x-axis.

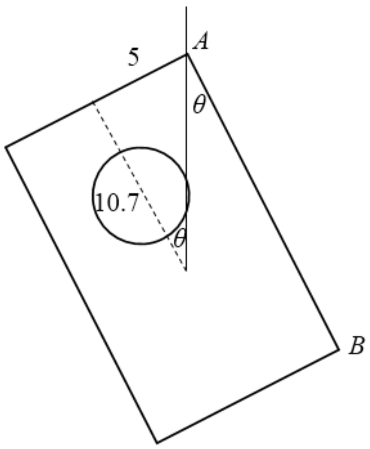
$$(200-9\pi)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=200\begin{pmatrix}10\\5\end{pmatrix}-9\pi\begin{pmatrix}6\\5\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=\frac{1}{200-9\pi}\begin{pmatrix}2000-54\pi\\1000-45\pi\end{pmatrix}$$

$$=\begin{pmatrix}10.658...\\5\end{pmatrix}$$

Therefore the centre of mass lies 10.7 cm from AD.

b

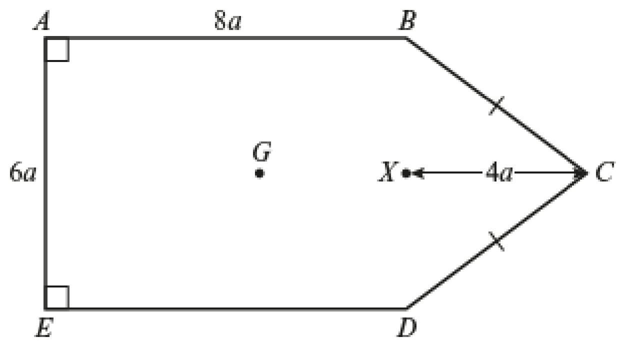


$$\tan\theta=\frac{5}{10.658...}$$

$$\theta=25.132...$$

$$=25^\circ \text{ (to the nearest degree)}$$

23 a



Let E be the origin and ED be the positive x-axis.

$$60a^2\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=48a^2\begin{pmatrix}4a\\3a\end{pmatrix}+12a^2\begin{pmatrix}\frac{28a}{3}\\3a\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=\frac{1}{60}\begin{pmatrix}304a\\180a\end{pmatrix}$$

$$=\begin{pmatrix}\frac{76}{15}a\\3a\end{pmatrix}$$

Therefore G lies $\frac{76}{15}a$ from AE

$$GX=8a-\frac{76}{15}a$$

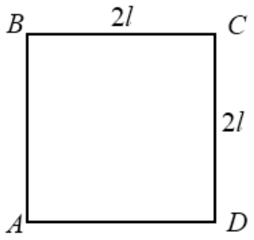
$$=\frac{44}{15}a \text{ as required.}$$

b Taking moments about the point of suspension gives:

$$M\times\frac{44}{15}a=\lambda M\times 4a$$

$$\lambda=\frac{11}{15}$$

24



Let A be the origin and let AD lie on the positive x-axis.

$$20M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=10M\begin{pmatrix}l\\ l\end{pmatrix}+M\begin{pmatrix}0\\ 0\end{pmatrix}+2M\begin{pmatrix}0\\ 2l\end{pmatrix}+3M\begin{pmatrix}2l\\ 2l\end{pmatrix}+4M\begin{pmatrix}2l\\ 0\end{pmatrix}$$

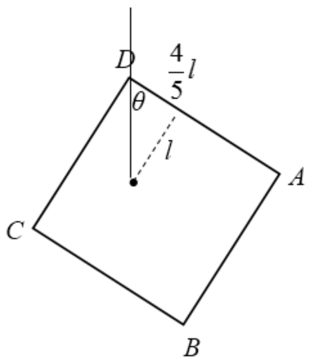
$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\frac{1}{20}\begin{pmatrix}24l\\ 20l\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}\frac{6}{5}l\\ l\end{pmatrix}$$

a The distance of the centre of mass from AB is $\frac{6}{5}l$

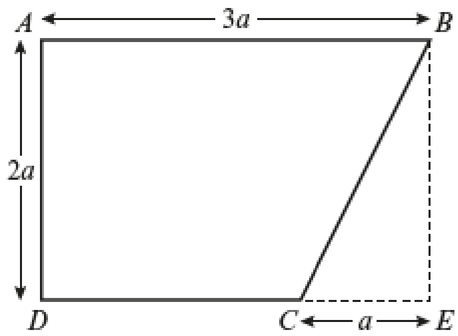
b The distance of the centre of mass from BC is l.

c



$$\tan\theta=\frac{l}{\frac{4}{5}l}$$
$$=\frac{5}{4}$$
$$\theta=51.340...$$
$$=51^\circ \text{ (to the nearest degree)}$$

25 a



$$5a^2\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=6a^2\begin{pmatrix}\frac{3}{2}a\\ a\end{pmatrix}-a^2\begin{pmatrix}\frac{8}{3}a\\ \frac{2}{3}a\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\frac{1}{5}\begin{pmatrix}\frac{19}{3}a\\ \frac{16}{3}a\end{pmatrix}$$

$$=\begin{pmatrix}\frac{19}{15}a\\ \frac{16}{15}a\end{pmatrix}$$

The centre of mass lies $\frac{19}{15}a$ from AD.

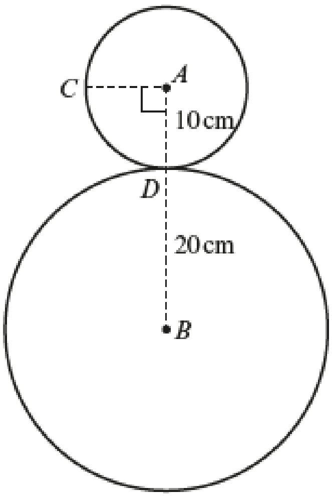
b Since AB is horizontal
Taking moments about the point of suspension gives:

$$\left(\frac{3}{2}a-\frac{19}{15}a\right)\times M=\frac{3}{2}a\times m$$

$$\frac{7}{30}M=\frac{3}{2}m$$

$$m=\frac{7}{45}M$$

26 a



Let the point B be the origin and let AB lie on the positive y-axis

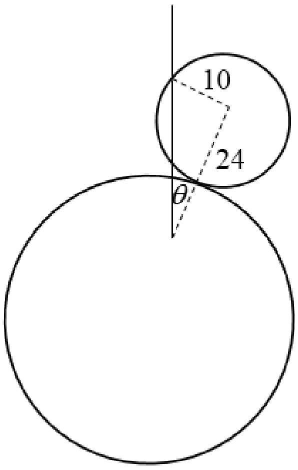
$$500\pi\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=400\pi\begin{pmatrix}0\\ 0\end{pmatrix}+100\pi\begin{pmatrix}0\\ 30\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\frac{1}{500\pi}\begin{pmatrix}0\\ 3000\pi\end{pmatrix}$$

$$=\begin{pmatrix}0\\ 6\end{pmatrix}$$

Therefore the centre of mass lies 6 cm from B.

b

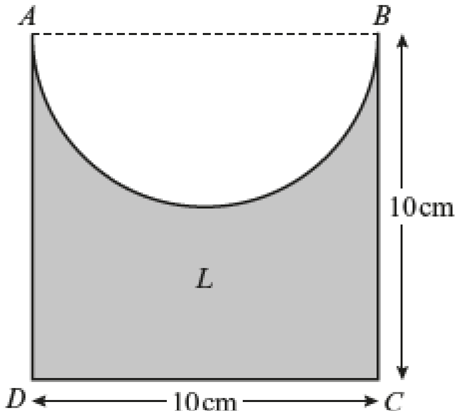


$$\tan\theta=\frac{10}{24}$$

$$\theta=22.619...$$

$$=22.6° \text{ (to 1 d.p.)}$$

27 a



Let A be the origin and let AB lie on the positive x-axis.

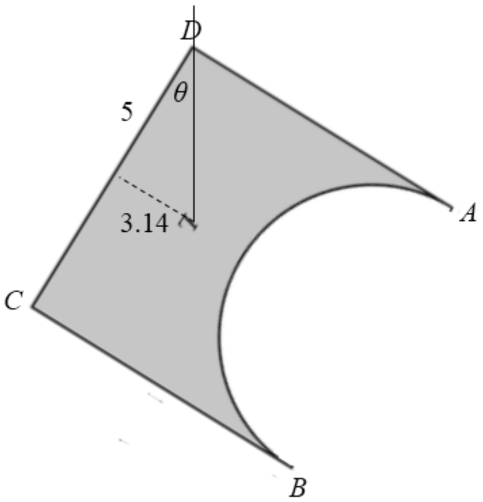
$$\left(100-\frac{25}{2}\pi\right)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=100\begin{pmatrix}5\\ -5\end{pmatrix}-\frac{25}{2}\pi\begin{pmatrix}5\\ -\frac{20}{3\pi}\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\frac{1}{\left(100-\frac{25}{2}\pi\right)}\begin{pmatrix}500-\frac{125}{2}\pi\\ -500+\frac{500}{6}\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}5\\ -6.860...\end{pmatrix}$$

Therefore the centre of mass lies 6.86 cm below AB.

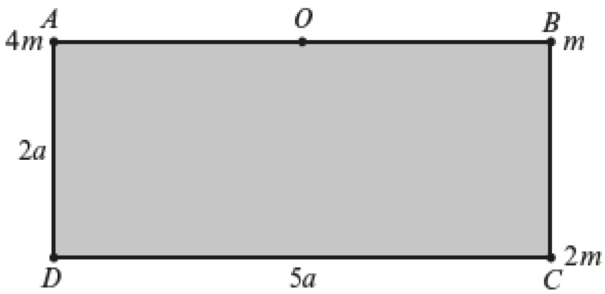
b



$$\tan\theta=\frac{3.1390...}{5}$$

$$\theta=32.1209...$$
$$=32.1° \text{ (to 1 d.p.)}$$

28 a



Let D be the origin and let DC lie on the x-axis.

$$10m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 3m\begin{pmatrix}2.5a\\ a\end{pmatrix} + 4m\begin{pmatrix}0\\ 2a\end{pmatrix} + m\begin{pmatrix}5a\\ 2a\end{pmatrix} + 2m\begin{pmatrix}5a\\ 0\end{pmatrix}$$

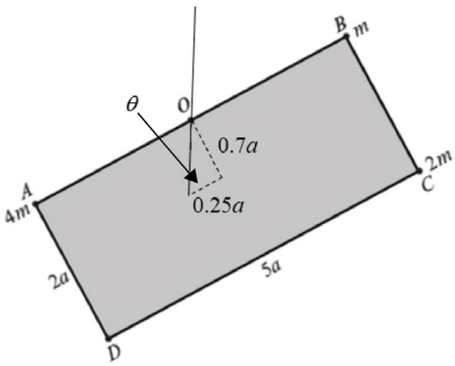
$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{10}\begin{pmatrix}22.5a\\ 13a\end{pmatrix}$$

$$= \begin{pmatrix}2.25a\\ 1.3a\end{pmatrix}$$

Therefore the centre of mass lies $2.25a$ from AD as required.

b The centre of mass lies $0.7a$ from AB.

c



$$\tan\theta = \frac{0.7a}{0.25a}$$

$\theta = 70.346... = 70°$ (to the nearest degree)

d Taking moments about O gives:

$$P \times 2a = 10mg \times (2.5a - \bar{x})$$

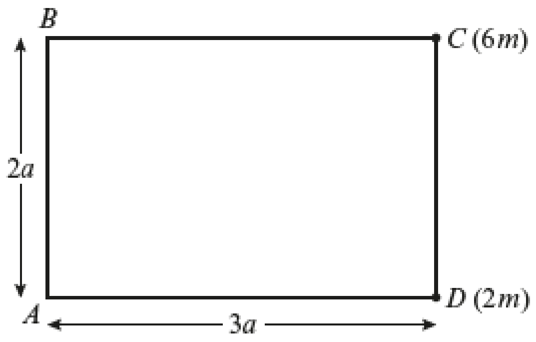
$$P = \frac{10mg \times (2.5a - 2.25a)}{2a}$$

$$= \frac{5}{4}mg \text{ as required}$$

e Magnitude of force $= \sqrt{(10mg)^2 + \left(\frac{5}{4}mg\right)^2}$

$$= \frac{5\sqrt{65}}{4}mg\text{N}$$

29 a



Let A be the origin and let AD be the positive x-axis.

$$12m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=m\begin{pmatrix}0\\ a\end{pmatrix}+m\begin{pmatrix}1.5a\\ 2a\end{pmatrix}+6m\begin{pmatrix}3a\\ 2a\end{pmatrix}+m\begin{pmatrix}3a\\ a\end{pmatrix}+2m\begin{pmatrix}3a\\ 0\end{pmatrix}+m\begin{pmatrix}1.5a\\ 0\end{pmatrix}$$

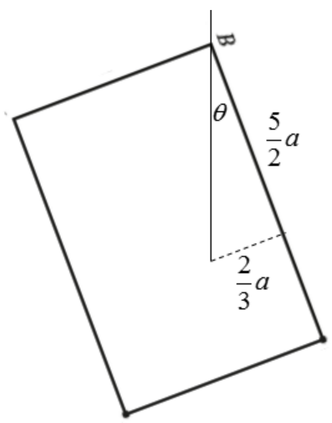
$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\frac{1}{12}\begin{pmatrix}30a\\ 16a\end{pmatrix}$$

$$=\begin{pmatrix}\frac{5}{2}a\\ \frac{4}{3}a\end{pmatrix}$$

i Therefore the centre of mass lies $\frac{5}{2}a$ from AB.

ii Therefore the centre of mass lies $\frac{4}{3}a$ from AD.

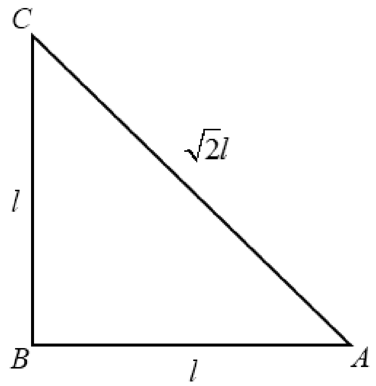
b



$$\tan\theta=\frac{\frac{2}{3}a}{\frac{5}{2}a}$$

$$\theta=14.931...$$
$$=14.9° \text{ (3 s.f.)}$$

30 a



Let A be the origin and let AB be the positive x-axis.

$$6m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 2m\begin{pmatrix}\frac{1}{2}l\\ 0\end{pmatrix} + m\begin{pmatrix}0\\ \frac{1}{2}l\end{pmatrix} + 3m\begin{pmatrix}\frac{1}{2}l\\ \frac{1}{2}l\end{pmatrix}$$

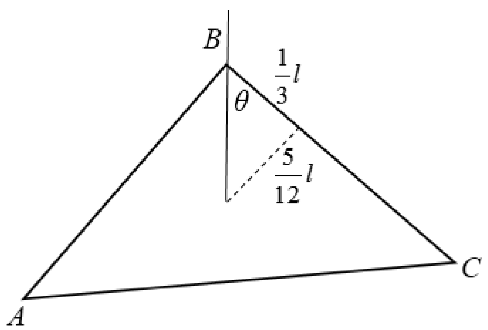
$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{6}\begin{pmatrix}\frac{5}{2}l\\ 2l\end{pmatrix}$$

$$= \begin{pmatrix}\frac{5}{12}l\\ \frac{1}{3}l\end{pmatrix}$$

i Therefore the centre of mass lies $\frac{5}{12}l$ from BC.

ii Therefore the centre of mass lies $\frac{1}{3}l$ from BA.

b

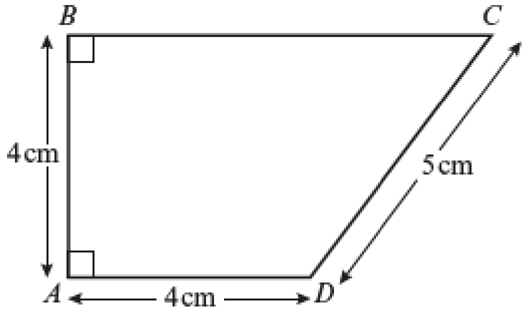


$$\tan\theta = \frac{\frac{5}{12}l}{\frac{1}{3}l}$$

$$\theta = 51.340...$$

$$= 50° \text{ (to the nearest degree)}$$

31 a



Let A be the origin and let AB be the positive x-axis.

$$0.225M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 0.04M\begin{pmatrix}0\\ 2\end{pmatrix} + 0.07M\begin{pmatrix}3.5\\ 4\end{pmatrix} + 0.075M\begin{pmatrix}5.5\\ 2\end{pmatrix} + 0.04M\begin{pmatrix}2\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{0.225}\begin{pmatrix}0.7375\\ 0.51\end{pmatrix}$$

$$= \begin{pmatrix}\frac{59}{18}\\ \frac{34}{15}\end{pmatrix}$$

Therefore the centre of mass lies $\frac{59}{18}$ from AB.

b Taking moments about the point of suspension gives:

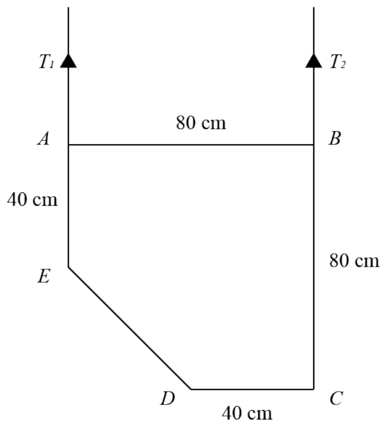
$$(3.5 - \bar{x}) \times M = 3.5 \times kM$$

$$k = \frac{\left(3.5 - \frac{59}{18}\right)}{3.5}$$

$$= \frac{4}{63}$$

Challenge

1 a



Let A be the origin and let AB be the positive x-axis.

$$5600\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 6400\begin{pmatrix}40\\ -40\end{pmatrix} - 800\begin{pmatrix}\frac{80}{3}\\ -\frac{200}{3}\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{25600}\begin{pmatrix}\frac{704000}{3}\\ -\frac{608000}{3}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{880}{21}\\ -\frac{760}{21}\end{pmatrix}$$

Therefore the centre of mass lies $\frac{880}{21}$ from AE.

b Res(↑) $T_1 + T_2 = W$ (1)

Taking moments about the centre of mass gives:

$$\frac{880}{21} \times T_1 = \left(80 - \frac{880}{21}\right) \times T_2$$

$$\frac{880}{21} T_1 = \frac{800}{21} T_2$$

$$T_1 = \frac{10}{11} T_2 \qquad (2)$$

Substituting (2) into (1) gives:

$$\frac{10}{11} T_2 + T_2 = W$$

$$\frac{21}{11} T_2 = W$$

$$T_2 = \frac{11}{21} W \text{ N and } T_1 = \frac{10}{21} W \text{ N}$$

1 **c** Res(↑) $T_1 + T_2 = W + kW$ (3)

Taking moments about the centre of mass gives:

$$\frac{880}{21} \times T_1 = \left(80 - \frac{880}{21}\right) \times T_2 + \left(80 - \frac{880}{21}\right) kW$$

$$\frac{880}{21} T_1 = \frac{800}{21} T_2 + \frac{800}{21} kW$$

$$T_1 = \frac{10}{11}\left(T_2 + kW\right) \qquad (4)$$

Substituting (4) into (3) gives:

$$\frac{10}{11}\left(T_2 + kW\right) + T_2 = W + kW$$

$$\frac{10}{11} T_2 + \frac{10}{11} kW + T_2 = W + kW$$

$$\frac{21}{11} T_2 = W + \frac{1}{11} kW$$

$$T_2 = \frac{11}{21} W + \frac{1}{21} kW$$

$$T_2 = \frac{1}{21} W\left(11 + k\right)$$

If T_2 exceeds $8W$ N it will snap, therefore:

$$\frac{1}{21} W\left(11 + k\right) < 8W$$

$$11 + k < 168$$

$$k < 157$$

If $T_2 = \frac{1}{21} W\left(11 + k\right)$ then substituting into (4) gives:

$$T_1 = \frac{10}{11}\left(\left(\frac{1}{21} W\left(11 + k\right)\right) + kW\right)$$

$$= \frac{10}{11}\left(\left(\frac{11}{21} W + \frac{1}{21} kW\right) + kW\right)$$

$$= \frac{10}{11}\left(\frac{11}{21} W + \frac{22}{21} kW\right)$$

$$= \frac{10}{21} W + \frac{20}{21} kW$$

If T_1 exceeds $10W$ N it will snap, therefore:

$$\frac{10}{21} W + \frac{20}{21} kW < 10W$$

$$10 + 20k < 210$$

$$k < 10$$

Largest value of k is 10

2 $v = 3\sin kt + \cos kt$, $t \geq 0$

$s = \int (3\sin kt + \cos kt)\,\mathrm{d}t$

$s = -\dfrac{3}{k}\cos kt + \dfrac{1}{k}\sin kt + c$ (1)

$\dfrac{\mathrm{d}v}{\mathrm{d}t} = 3k\cos kt - k\sin kt$

At $t = 0$, $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 1.5$

$3k\cos k(0) - k\sin k(0) = 1.5$

$3k = 1.5$

$k = 0.5$

Substituting $k = 0.5$ into (1) gives:

$s = -\dfrac{3}{(0.5)}\cos(0.5)t + \dfrac{1}{(0.5)}\sin(0.5)t + c$

$s = -6\cos(0.5t) + 2\sin(0.5t) + c$

When $t = 0$, $s = 0$

$(0) = -6\cos(0) + 2\sin(0) + c$

$c = 6$

Therefore:

$s = -6\cos(0.5t) + 2\sin(0.5t) + 6$

$\dfrac{\mathrm{d}s}{\mathrm{d}t} = 3\sin(0.5t) + \cos(0.5t)$

At maximum value $\dfrac{\mathrm{d}s}{\mathrm{d}t} = 0$

$3\sin(0.5t) + \cos(0.5t) = 0$

$3\sin(0.5t) = -\cos(0.5t) = 0$

$\dfrac{\sin(0.5t)}{\cos(0.5t)} = -\dfrac{1}{3}$

$\tan(0.5t) = -\dfrac{1}{3}$

$0.5t = 161.565...$

$t = 323.130...$

$= 323$ s (3 s.f.)

When t = 323.130…

$s = -6\cos(0.5(323.130...)) + 2\sin(0.5(323.130...)) + 6$

$= 12.324...$

$= 12.3$ m (3 s.f.)

3 Res(→) $d\cos\theta = ut\sin\theta$

$$t = \frac{d\cos\theta}{u\sin\theta}$$

$$t = \frac{d}{u\tan\theta} \quad (1)$$

Res(↑) $-d\sin\theta = ut\cos\theta - \frac{1}{2}gt^2 \quad (2)$

Substituting (1) into (2) gives:

$$-d\sin\theta = u\left(\frac{d}{u\tan\theta}\right)\cos\theta - \frac{1}{2}g\left(\frac{d}{u\tan\theta}\right)^2$$

$$-d\sin\theta = \frac{d\cos\theta}{\tan\theta} - \frac{gd^2}{2u^2\tan^2\theta}$$

$$\frac{d\cos\theta}{\tan\theta} + d\sin\theta - \frac{gd^2}{2u^2\tan^2\theta} = 0$$

$$d\left(\frac{\cos\theta}{\tan\theta} + \sin\theta\right) - \frac{gd^2}{2u^2\tan^2\theta} = 0$$

$$d\left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta}\right) - \frac{gd^2}{2u^2\tan^2\theta} = 0$$

$$d\left(\frac{1}{\sin\theta}\right) - \frac{gd^2}{2u^2\tan^2\theta} = 0$$

$$d - \frac{\frac{gd^2}{2u^2\tan^2\theta}}{\frac{1}{\sin\theta}} = 0$$

$$d - \frac{gd^2\sin\theta}{2u^2\tan^2\theta} = 0$$

$$d - \frac{gd^2\sin\theta\cos^2\theta}{2u^2\sin^2\theta} = 0$$

$$d - \frac{gd^2\cos^2\theta}{2u^2\sin\theta} = 0$$

$$d\left(1 - \frac{gd\cos^2\theta}{2u^2\sin\theta}\right) = 0$$

$$d = 0 \text{ or } 1 - \frac{gd\cos^2\theta}{2u^2\sin\theta} = 0$$

$$\frac{gd\cos^2\theta}{2u^2\sin\theta} = 1$$

$$d = \frac{2u^2\sin\theta}{g\cos^2\theta}$$

$$d = \frac{2u^2}{g}\tan\theta\sec\theta \text{ as required.}$$